American University of Beirut
Department of Computer Science
CMPS 211 – Discrete Mathematics – Fall 15/16
Assignment 5

Please solve the following exercises and submit **BEFORE 11:55 pm of Monday 2nd of November.**

Please get a hardcopy submission whether you plan to solve it on a computer or on a paper. In Both cases, also submit to Moodle. However, if you submit a handwritten solution, I will only correct the questions that I manage to read (and easily find).

You can submit the hard copies during by Wednesday during the recitation. I will use Moodle submissions as a proof of early submissions. Don't try to modify anything in the hard copy submission, or else it will be considered cheating and you'll get a Zero.

Exercise 1 (10 points)

Can you guess the next number?

2 6 18 54 162 ...

a) Show that the value of the nth element $a_n = 3^n - 3^{n-1}$

$$a_n = 3^{n-1}(3-1) = 3^{n-1} * 2$$
$$a_1 = 3^{1-1} * 2 = 1 * 2 = 2$$
$$a_2 = 3^{2-1} * 2 = 3 * 2 = 6$$
$$a_3 = 3^{3-1} * 2 = 9 * 2 = 18$$

b) Find an equation for $S_n = a_1 + a_2 + a_3 + .. + a_i + \cdots + a_n$

$$S_n = a_1 + a_2 + a_3 + .. + a_i + \cdots + a_n$$
$$S_n = 3^n - 1$$

c) Prove the formula you conjectured in part (b).

Basic Step:

$$S_1 = 3^1 - 1 = 2, \text{Valid}$$
$$S_2 = 3^2 - 1 = 8, \text{Valid}$$

Inductive Step:

$$S_n = 3^n - 1, \text{then } S_{n+1} = 3^{n+1} - 1$$
$$S_{n+1} = a_{n+1} + S_n$$

$S_{n+1} = 3^{n+1} - 3^n + 3^n - 1 = 3^{n+1} - 1$ Proved

Exercise 2 (10 points)

Prove that 6 divides 3^n - 3 whenever n is an integer > 0.

Basic Step:

Consider f(n) = 3^n-3,
f(1) = 0, which is divisible by 6
f(2) = 9 – 3 = 6, which is divisible by 6
f(3) = 27 – 3 = 24, which is divisible by 6

Inductive step:

f(n) = 3^n – 3 is divisible by 6, then f(n+1) is divisible by 6
f(n+1) = 3^{n+1}-3
= 3*(3^n) – 3
= 3*3^n – 9 + 6
= 3*(3^n – 3) + 6
= 3*f(n) + 6
= 3 * 6k + 6, since f(n) is divisible by 6
= 6 (3k + 1), which is divisible by 6
Proved by Induction!

It can be also proved without induction by saying that 3^n-3 is 3(3^{n-1} – 1), and 3^{n-1} – 1 is always divisible by 2 since 3^x is always odd for any positive integer x, and thus 3^n-3 is divisible by 3 and 2 and then it is divisible by 6

Exercise 3 (10 points)

What is wrong with this "proof"?

- "Theorem" For every positive integer n, $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$.
- Then $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$.

- By this inductive hypothesis, $\sum_{i=1}^{k+1} i = \frac{\left(k+\frac{1}{2}\right)^2}{2} + k + 1 = \frac{\left(k^2+k+\frac{1}{4}\right)}{2} + k + 1 = \frac{\left(k^2+3k+\frac{9}{4}\right)}{2} = \frac{\left(k+\frac{3}{2}\right)^2}{2} = \frac{\left[(k+1)+\frac{1}{2}\right]^2}{2}$, completing the inductive step.

For $n = 1$, $\sum_{i=1}^{n} i = 1$, and $\frac{\left(1+\frac{1}{2}\right)^2}{2} = \frac{2.25}{2}$, and thus the basic step doesn't hold

Exercise 4 (10 points)

Suppose that m and n are positive integers with m > n and f is a function from {1,2,...,m} to {1,2,...,n}. Use mathematical induction on the variable n to show that f is not one-to-one. [***Hint:*** *apply induction on n]*

Basic Step:
For n = 1 and any value of m > n, then f maps from {1, 2, ..m} to {1}, then multiple values in domain maps to {1}, and thus f is not one-to-one

Inductive step:
For any arbitrary n and m, such that m > n, f is not one-to-one
For n+1, such that n+1 < m, f maps from {1, 2, ..., m} to {1, 2, ..., n, n+1}:

- If no value in domain maps to n+1, then f maps {1, 2, ..., m} to {1, 2, ..., n} is not one to one by inductive hypothesis.
- If some value in {1, 2, ..., m} maps to n+1, call it *i*, then *i* maps to *n+1*, then we can remove "swap" *i* by *m* and remove *m* from the domain; so {1, 2, ... m-1} maps to {1, 2, ...n}, and m-1 > n since (n+1 < m), which is not one-to-one also by inductive hypothesis

Then f is not one-to-one

Exercise 5 (10 points)

In computer science, a binary tree is a tree data structure in which each node has at most two children, which are referred to as the left child and the right child (https://en.wikipedia.org/wiki/Binary_tree).

A Ternary tree is similar to a binary tree, however instead of 2 children, each node can have up to 3 children.

A prefect Tree is a tree such that all leaf nodes are of same depth, and all other nodes are full nodes; i.e: each node in a perfect ternary tree of depth h has 3 children, except for the nodes at depth h (leaf nodes) who have 0 children.

a) Formulate a conjecture about the number of nodes in a Perfect Ternary tree. You may assume that the smallest perfect Ternary tree has 1 single node, and height 0

For h = 0, total number of nodes in a ternary tree is 1
For h = 1, total number of nodes in a ternary tree is 4
For h = 2, total number of nodes in a ternary tree is 13
We can say that for an arbitrary height h, total number of nodes is

$$\text{nodes(h)} = \sum_{i=0}^{h} 3^{i} = \frac{1-3^{h+1}}{1-3} = \frac{3^{h+1}-1}{2}$$

b) Prove it using induction.

Inductive step:
Any perfect ternary tree of height h can be replicated 3 times, and joined by a common root, to create a perfect ternary tree (since the 3 subtrees of the root are subtrees) of height h+1

Based on inductive hypothesis, $\text{nodes}(h+1) = \frac{3^{h+2}-1}{2}$

Since each subtree is of height h, then the total number of nodes in each is nodes(h), thus the total number of nodes in a tree of height h+1 is:
$\text{nodes}(h+1) = 3 * \text{nodes}(h) + 1$, (1 is the root node)

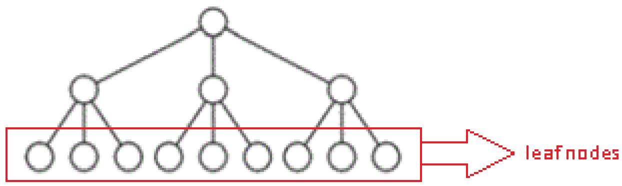
$$\text{nodes}(h+1) = 3 * \frac{3^{h+1}-1}{2} + 1$$

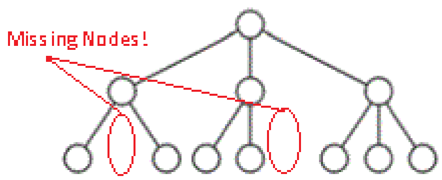
$$= \frac{3^{h+2}-3}{2} + 1$$

$$= \frac{3^{h+2}-1-2}{2} + 1$$

$$= \frac{3^{h+2}-1}{2} + 1 - 1 = \frac{3^{h+2}-1}{2}$$

Then the total number of nodes in a perfect ternary tree is $\frac{3^{h+1}-1}{2}$. Proved by Induction





Exercise 6 (10 points)

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar can be broken along a vertical or a horizontal line separating the squares to get 2 rectangular pieces. Assuming that only one piece can be broken at a time:

a) Determine how many breaks you must successively make to break the bar into n separate squares
if n = 1, you need 0 breaks
if n = 2, you need 1 break
if n = 3, you need 2 breaks
if n = 4, you need 3 breaks.
So for any bar of n pieces, we need n-1 breaks.
B(n) = n-1

b) Use strong induction to prove your answer
Basic step:

Shown in part a.

for all j / 0 < j <= n, B(n) = n-1

Inductive step:

Given a bar of n+1 pieces *[we expect B(n+1) = n]*, cutting it once forms 2 bars of size a and b, such that a+b = n+1, a & b >= 1, and a & b are integers; and thus 0 < a <= n, and 0 < b <= n.

Thus by inductive hypothesis, B(a) = a-1, and B(b) = b-1

Now B(n) = B(a) + B(b) + 1 (+1 is the first cut)

B(n+1) = a-1 + b-1 + 1 = a + b -1 = n;

Exercise 7 (10 points)

Consider the proposition $2^n > n^3$

a) Find an integer N such that the proposition is true whenever n is greater than N.

For n = 1, $2^1 > 1^3$, is false
For n = 2, $2^2 > 2^3$, is false
For n = 3, $2^3 > 3^3$, is false
For n = 4, $2^4 > 4^3$, is false
For n = 5, $2^5 > 5^3$, is false
For n = 6, $2^6 > 6^3$, is false
For n = 7, $2^7 > 7^3$, is false
For n = 8, $2^8 > 8^3$, is false
For n = 9, $2^9 > 9^3$, is false
For n = 10, $2^{10} > 10^3$, is 1024 > 1000, which is true
And thus for all n >= 10, the proposition is true

b) Prove that your result for all n > N using mathematical induction.

Basic step:

Shown in part a.

$P(n) = 2^n > n^3$, is true for all n >= 10

Inductive Step:

$$P(n+1) = 2^{n+1} > (n+1)^3 \ ?$$
$$2^{n+1} = 2 * 2^n > 2n^3, \text{using inductive hypothesis}$$
$$\leftrightarrow 2^{n+1} > 2n^3$$
$$\leftrightarrow 2^{n+1} > n^3 + n^3$$
$$\leftrightarrow 2^{n+1} > n^3 + 6n^2, \text{since n >=10}$$
$$\leftrightarrow 2^{n+1} > n^3 + 3n^2 + 10n, \text{since n >=10}$$
$$\leftrightarrow 2^{n+1} > n^3 + 3n^2 + 3n + 7n$$
$$\leftrightarrow 2^{n+1} > n^3 + 3n^2 + 3n + 1$$
$$\leftrightarrow 2^{n+1} > (n+1)^3$$

OR

$$2^{n+1} = 2 * 2^n > 2n^3$$
$$\text{Is } 2n^3 > (n+1)^3?$$

Basic steps for n >=10 works

Inductive step:

$$2(n+1)^3 > (n+2)^3?$$
$$2(n+1)^3 = 2n^3 + 6n^2 + 6n + 2$$
$$(n+2)^3 = n^3 + 6n^2 + 12n + 8$$

$$2(n+1)^3 - (n+2)^3 = n^3 - 6n - 6 = n(n^2 - 6) - 6 >?\ 0$$
$$n \geq 10, then\ n^2 - 6 \geq 96$$
$$n(n^2 - 6) - 6 \geq 0$$

Then

$$2(n+1)^3 > (n+2)^3$$
$$\text{So } 2^{n+1} > 2n^3 > (n+1)^3$$

Exercise 8 (10 points)

Assume you can only use 5-cent and 9-cent stamps.

a) Determine which amounts of postage can be formed by the given stamps
We need n = 5a + 9b such that a, b >= 0, and a and b are integers
Thus we can form postage of value 5, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, 30**, 32, 33, 34, 35, 36, 37, 38, 39, 40…**

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.
Basic step:
Using stamps of 5 and 9 cents only, we were able to make postages with amounts = **32, 33, 34, 35, 36, 37, 38, 39, 40…** as shown in part a

Inductive step:
Using stamps of 5 and 9 cents, we can create postage of amount n >= 32
In other words, n = 5a + 9b for n >=32, a,b >= 0 and a, b & n are integers.
So by inductive hypothesis, there is a configuration of a and b such that the above conditions are true, then n = 5a + 9b, for some value of n >=32.

We need to show that we can form n+1, such that n+1 = 5a' + 9b', with conditions for a' and b' similar to those of a and b

$$n + 1 = \begin{cases} 5(a+2) + 9(b-1) \\ 5(a-7) + 9\,(b+4) \end{cases}$$

then [a' = a+2 and b' = b-1], or [a' =a-7 and b'=b+4]
notice that 5a' + 9b' in both cases is equals to n+1

Now case 1: if n = 5a + 9b has b >=1, then b' = b-1 >=0, then b' is valid; hence we can form n+1 = 5(a+2) + 9(b-1); i.e: by adding 2 5-cent stamps, and removing 1 9-cent stamp.
The other case would be that n = 5a + 9b and b = 0, then n = 5a, we know that n>= 32, then 5a >= 32, then a >= 32/5 → a >= 6.4, but a is an integer, so a >= 7, then a-7 >= 0, then a'= a-7 is also valid; hence we can form n+1 = 5(a-7) + 9(b+4); i.e: by removing 7 5-cents stamps, and adding 6 9-cent stamps
Then we can get n+1 if we have n formed of 5-cents and 9-cents stamps and n >= 32
Proved by induction.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?
We know from basic step shown above that the there is a configuration of a and b for n = 32, 33, 34, 35, and 36.
So for all j/ 32<= j <= n, and 36 <= n the conjecture above holds.

To get n+1, we simply add a 5-cents stamp to the configuration of n-4, giving a total of n+1 cents. We know that n-4 has a valid configuration by inductive hypothesis since 36 <= n, then 32 <= n -4 <= n (j = n-4)
So, since n-4 can be made of 5-cents and 9-cents stamps, adding a 5-cent stamp will give a total of n+1 cents made of 5 and 9 cents stamps also.
Proved by Strong Induction.!

Exercise 9 (10 points)

Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on.
*[**Hint:** For the inductive step, separately consider the case where k + 1 is even and where it is odd. When it is even, note that (k + 1)/2 is an integer.]*

Basic Step:
$1 = 2^0$, this is a sum of distinct powers of 2
$2 = 2^1$, this is a sum of distinct powers of 2
$3 = 2^0 + 2^1$, this is a sum of distinct powers of 2
$4 = 2^2$, this is a sum of distinct powers of 2

Inductive step:
for all j/ 1<= j <= k, for an arbitrary k, j can be written as a sum of distinct powers of 2
We need to show that this property holds for k+1
If k is even, then 2^0 isn't present in the sum since 2^0 is the only odd number in all powers of 2 and k is even, then we can form k+1 by adding 2^0 to the sum of distinct powers, keeping them distinct
The other case would be that k is odd, in which it would be the case that k+1 is even, and (k+1)/2 is an integer in the range 1<= (k+1)/2 <= k, then (k+1)/2 is a sum distinct powers of 2 by inductive hypothesis. Adding 1 to each power of 2 (i.e: left shifting the powers) will keep the powers of 2 distinct and will give us 2*(k+1)/2, which is k+1
Therefore, the predicate is proved by Strong induction